

Vector space model

September 22, 2024

Reading material: [Chapter 6](#) of IIR

Overview

- 1 Why ranked retrieval?
- 2 Term frequency
- 3 tf-idf weighting
- 4 The vector space model

Take-away today

- **Ranking** search results: why it is important (as opposed to just presenting a set of unordered Boolean results)
- **Term frequency**: This is a key ingredient for ranking.
- **Tf-idf ranking**: best known traditional ranking scheme
- **Vector space model**: Important formal model for information retrieval (along with Boolean and probabilistic models)

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Ranked retrieval

- Thus far, our queries have been **Boolean**.
 - Documents either match or don't.
- **Good for expert users** with precise understanding of their needs and of the collection.
- Also **good for applications**: Applications can easily consume 1000s of results.
- **Not good for the majority of users**
- Most users are not capable of writing Boolean queries ...
 - ...or they are, but they think it's too much work.
- Most users don't want to wade through 1000s of results.
- This is particularly true of web search.

Problem with Boolean search: Feast or famine

- Boolean queries often result in either too few ($=0$) or too many (1000s) results.
- Query 1 (boolean conjunction): [standard user dlink 650]
 - $\rightarrow$ 200,000 hits – [feast](#)
- Query 2 (boolean conjunction): [standard user dlink 650 no card found]
 - $\rightarrow$ 0 hits – [famine](#)
- In Boolean retrieval, it takes a lot of skill to come up with a query that produces a manageable number of hits.

Feast or famine: No problem in ranked retrieval

- With ranking, large result sets are not an issue.
- Just show the top 10 results
- Doesn't overwhelm the user
- Premise: the ranking algorithm works: More relevant results are ranked higher than less relevant results.

Scoring as the basis of ranked retrieval

- How can we accomplish a **relevance ranking** of the documents with respect to a query?
- Assign a score to each query-document pair, say in $[0, 1]$.
- This score measures how well document and query “match”.
- Sort documents according to scores

Query-document matching scores

- How do we compute the score of a query-document pair?
- Intuitively
 - no query term occurs in document $\rightarrow$ score ≈ 0 .
 - more frequent a query term in document $\rightarrow$ higher score
 - more query terms in document $\rightarrow$ higher score
- How to formalize it?

Take 1: Jaccard coefficient

- A commonly used measure of overlap of two sets
- Let A and B be two sets
- Jaccard coefficient:

$$Jaccard(A, B) = \frac{|A \cap B|}{|A \cup B|} \quad A \neq \emptyset \quad B \neq \emptyset \quad (1)$$

- $JACCARD(A, A) = 1$
- $JACCARD(A, B) = 0$ if $|A \cap B| = 0$
- A and B don't have to be the same size.
- Always assigns a number between 0 and 1.

Jaccard coefficient: Example

- Jaccard coefficient for:
 - Query: “ides of March”
 - Document “Caesar died in March”
- $\text{JACCARD}(q, d) = 1/6$

$$\text{jaccard}(A, B) = \frac{|A \cap B|}{|A \cup B|} \quad (2)$$

What's wrong with Jaccard Similarity?

- Term frequency: It doesn't consider *term frequency* (how many occurrences a term has).
 - Query: "ides of March"
 - Document "Caesar died in March. In March.."It does not rank this doc higher.
- Document frequency: it does not consider document frequency (how many times it occurs in all the documents).
 - Rare terms are more informative than frequent terms. Jaccard similarity does not consider this information.
- The way to normalize the length of the document
 - we'll use $|A \cap B| / \sqrt{\text{Length}(A)\text{Length}(B)}$ (cosine) ...
 - ...instead of $|A \cap B| / |A \cup B|$ (Jaccard) for length normalization.

tfidf caricature analogy

- TF: accentuates terms that are frequent in the document,
- IDF: but not frequent in general



- Caricature: exaggerates traits that are characteristic of the person (compared to the average)

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Binary incidence matrix

	Anthony and Cleopatra	Julius Caesar	The Tempest	Hamlet	Othello	Macbeth	...
ANTHONY	1	1	0	0	0	1	
BRUTUS	1	1	0	1	0	0	
CAESAR	1	1	0	1	1	1	
CALPURNIA	0	1	0	0	0	0	
CLEOPATRA	1	0	0	0	0	0	
MERCY	1	0	1	1	1	1	
WORSER	1	0	1	1	1	0	

...

Each document is represented as a binary vector $\in \{0, 1\}^{|M|}$.

Count matrix

	Anthony and Cleopatra	Julius Caesar	The Tempest	Hamlet	Othello	Macbeth	...
ANTHONY	157	73	0	0	0	1	
BRUTUS	4	157	0	2	0	0	
CAESAR	232	227	0	2	1	0	
CALPURNIA	0	10	0	0	0	0	
CLEOPATRA	57	0	0	0	0	0	
MERCY	2	0	3	8	5	8	
WORSER	2	0	1	1	1	5	

...

Each document is now represented as a count vector $\in \mathbb{N}^{|V|}$.

Bag of words model

- We do not consider the **order** of words in a document.
 - *John is quicker than Mary* and
 - *Mary is quicker than John*
 - are represented the same way.
- This is called a **bag of words model**.
- In a sense, this is a step back: The positional index was able to distinguish these two documents.
- positional information can be recovered.
- For now: bag of words model

Term frequency tf

- The (raw) term frequency $tf_{t,d}$ of term t in document d is defined as the number of times that t occurs in d .
- Raw term frequency is not what we want because:
 - A document with $tf = 10$ occurrences of the term is more relevant than a document with $tf = 1$ occurrence of the term.
 - But not 10 times more relevant.
 - Relevance does not increase proportionally with term frequency.

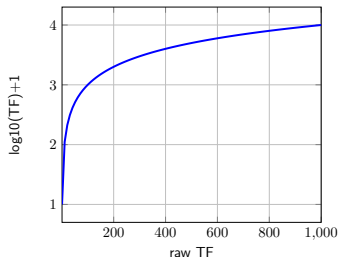
Instead of raw frequency: Log frequency weighting

- The log frequency weight of term t in d :

$$w_{t,d} = \begin{cases} 1 + \log_{10} \text{tf}_{t,d} & \text{if } \text{tf}_{t,d} > 0 \\ 0 & \text{otherwise} \end{cases}$$

- $\text{tf}_{t,d} \rightarrow w_{t,d}$:

$0 \rightarrow 0, 1 \rightarrow 1, 2 \rightarrow 1.3, 10 \rightarrow 2, 1000 \rightarrow 4$, etc.



other tf weight

$$TF_{t,d} = \begin{cases} c(t, d) & \text{RawTF} \\ 1 + \log(c(t, d)) & \text{LogTF} \\ 0.5 + \frac{c(t,d)}{2 \times \text{MaxFreqd}} & \text{Max freq normalization} \\ \frac{(k+1)c(t,d)}{c(t,d) + k \left(1 - b + \frac{b \times \text{docLen}}{\text{avgDocLen}} \right)} & \text{Okapi/BM25} \end{cases}$$

- k: a parameter of small value (around one),
- b: is a parameter taking a value between 0 and 1.
- BM: Best Matching

Why TF normalization

- doc len varies
- repeated occurrences are less informative than the first occurrence
- penalize long doc
- avoid over-penalizing (pivoted normalization)

$$1 - b + b \frac{docLen}{avgDocLen} \quad (3)$$

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Frequency in document vs. frequency in collection

- In addition to term frequency (the frequency of the term in the document) ...
- ...we also want to use the frequency of the term in the collection for weighting and ranking.

Desired weight for rare terms

- Rare terms are more informative than frequent terms.
- Consider a term in the query that is **rare** in the collection (e.g., ARACHNOCENTRIC).
- A document containing this term is very likely to be relevant.
- → We want **high weights for rare terms** like ARACHNOCENTRIC.

Desired weight for frequent terms

- Frequent terms are less informative than rare terms.
 - recall stop words.
- Consider a term in the query that is frequent in the collection (e.g., GOOD, INCREASE, LINE).
- A document containing this term is more likely to be relevant than a document that doesn't ...
- ...but words like GOOD, INCREASE and LINE are not sure indicators of relevance.
- Design rationale: frequent terms should have a lower weight
- How to formalize/quantify it?

Document frequency

- We want high weights for rare terms like ARACHNOCENTRIC.
- We want low (positive) weights for frequent words like GOOD, INCREASE, and LINE.
- We will use document frequency to factor this into computing the matching score.

document frequency(DF)

df_t of term t is the number of documents in the collection that t occurs in.

idf weight

- df_t is the document frequency, the number of documents that t occurs in.
- df_t is an inverse measure of the **informativeness** of term t .
- We define the **idf weight** of term t as follows:

$$idf_t = \log_{10} \frac{N}{df_t}$$

(N is the number of documents in the collection.)

- idf_t is a measure of the **informativeness** of the term.
- $\log N/df_t$ instead of N/df_t to “dampen” the effect of *idf*
- Note that we use the log transformation for both term frequency and document frequency.

Examples for idf

Compute idf_t using the formula: $\text{idf}_t = \log_{10} \frac{1,000,000}{\text{df}_t}$

term	df_t	idf_t
calpurnia	1	6
animal	100	4
sunday	1000	3
fly	10,000	2
under	100,000	1
the	1,000,000	0

- If a term occurs in every document, its weight is zero.
- It is the same as treating it as a stop word

Effect of idf on ranking

- idf affects the ranking of documents for queries with at least two terms.
- For example, in the query “arachnocentric line”, idf weighting increases the relative weight of ARACHNOCENTRIC and decreases the relative weight of LINE.
- idf has little effect on ranking for one-term queries.

Collection frequency vs. Document frequency

word	collection frequency	document frequency
INSURANCE	10440	3997
TRY	10422	8760

- Collection frequency of t : number of tokens of t in the collection
- Document frequency of t : number of documents t occurs in
- Why these numbers?
- Which word is a better search term (and should get a higher weight)?
- This example suggests that df (and idf) is better for weighting than cf (and “icf”).

tf-idf weighting

- The tf-idf weight of a term is the product of its tf weight and its idf weight.



$$w_{t,d} = (1 + \log \text{tf}_{t,d}) \cdot \log \frac{N}{\text{df}_t}$$

- tf-weight
- idf-weight
- Best known weighting scheme in information retrieval
- Alternative names: tf.idf, tf x idf

Summary: tf-idf

- Assign a tf-idf weight for each term t in each document d :
$$w_{t,d} = (1 + \log \text{tf}_{t,d}) \cdot \log \frac{N}{\text{df}_t}$$
- The tf-idf weight ...
 - ...increases with the number of occurrences within a document.
(term frequency)
 - ...increases with the rarity of the term in the collection.
(inverse document frequency)

Exercise: Term, collection and document frequency

Quantity	Symbol	Definition
term frequency	$tf_{t,d}$	number of occurrences of t in d
document frequency	df_t	number of documents in the collection that t occurs in
collection frequency	cf_t	total number of occurrences of t in the collection

- Relationship between df and cf ?
- Relationship between tf and cf ?
- Relationship between tf and df ?

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...

Each document is represented as a binary vector $\in \{0, 1\}^{|M|}$.

Count matrix

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MERCY	2	0	3	8	5	8	
WORSER	2	0	1	1	1	5	

...

Each document is now represented as a count vector $\in \mathbb{N}^{|V|}$.

Binary $\rightarrow$ count $\rightarrow$ weight matrix

	Anthony and Cleopatra	Julius Caesar	The Tempest	Hamlet	Othello	Macbeth	...
ANTHONY	5.25	3.18	0.0	0.0	0.0	0.35	
BRUTUS	1.21	6.10	0.0	1.0	0.0	0.0	
CAESAR	8.59	2.54	0.0	1.51	0.25	0.0	
CALPURNIA	0.0	1.54	0.0	0.0	0.0	0.0	
CLEOPATRA	2.85	0.0	0.0	0.0	0.0	0.0	
MERCY	1.51	0.0	1.90	0.12	5.25	0.88	
WORSER	1.37	0.0	0.11	4.15	0.25	1.95	

...

Each document is now represented as a real-valued vector of tf-idf weights

$$\in \mathbb{R}^{|M|}.$$

Documents as vectors

- Each document is now represented as a real-valued vector of tf-idf weights $\in \mathbb{R}^{|V|}$.
- So we have a $|V|$ -dimensional real-valued vector space.
- Terms are **axes** of the space.
- Documents are **points** or **vectors** in this space.
- Very high-dimensional: tens of millions of dimensions when you apply this to web search engines
- Each vector is very sparse - most entries are zero.
- Reduce the dimensionality will be one of our focuses.

Queries as vectors

- Key idea 1: do the same for queries: represent them as vectors in the high-dimensional space
- Key idea 2: Rank documents according to their proximity to the query
- proximity = similarity
- proximity $\approx$ negative distance
- Recall: We're doing this because we want to get away from the you're-either-in-or-out, feast-or-famine Boolean model.
- Instead: rank relevant documents higher than non-relevant documents

How do we formalize vector space similarity?

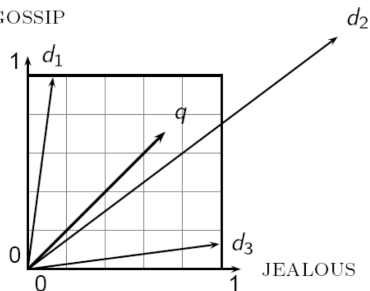
- First cut: (negative) distance between two points
- (= distance between the end points of the two vectors)
- Euclidean distance?
- Euclidean distance is a bad idea ...
- ...because Euclidean distance is **large** for vectors **of different lengths**.

Euclidean distance

- Euclidean distance between **q** and **d2** is large
- Even though the distribution of terms in the query **q** and the distribution of terms in the document **d2** are very similar.

term	d1	d2	d3	q
jealous	0.06	1.85	1.00	0.7
gossip	1.0	1.30	0.06	0.7

GOSSIP

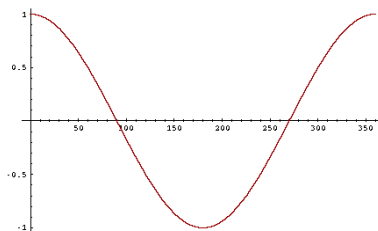


Use angle instead of distance

- Rank documents according to angle with query
- Thought experiment: take a document d and append it to itself. Call this document d' . d' is twice as long as d .
- “Semantically” d and d' have the same content.
- The angle between the two documents is 0, corresponding to maximal similarity ...
- ...even though the Euclidean distance between the two documents can be quite large.

From angles to cosines

- The following two notions are equivalent.
 - Rank documents according to the **angle** between query and document in decreasing order
 - Rank documents according to **cosine**(query,document) in increasing order
- Cosine is a monotonically decreasing function of the angle for the interval $[0^\circ, 180^\circ]$



Length normalization

- How do we compute the cosine?
- A vector can be (length-) normalized by dividing each of its components by its length – here we use the L_2 norm:

$$||x||_2 = \sqrt{\sum_i x_i^2}$$

- This maps vectors onto the unit sphere ...
- ...since after normalization: $||x||_2 = \sqrt{\sum_i x_i^2} = 1.0$
- As a result, longer documents and shorter documents have weights of the same order of magnitude.
- Effect on the two documents d and d' (d appended to itself) from earlier slide: they have **identical vectors** after length-normalization.

Cosine similarity between query and document

$$\cos(\vec{q}, \vec{d}) = \text{SIM}(\vec{q}, \vec{d}) = \frac{\vec{q} \cdot \vec{d}}{|\vec{q}| |\vec{d}|} = \frac{\sum_{i=1}^{|V|} q_i d_i}{\sqrt{\sum_{i=1}^{|V|} q_i^2} \sqrt{\sum_{i=1}^{|V|} d_i^2}}$$

- q_i is the tf-idf weight of term i in the query.
- d_i is the tf-idf weight of term i in the document.
- $|\vec{q}|$ and $|\vec{d}|$ are the lengths of $\vec{q}$ and $\vec{d}$.
- This is the **cosine similarity** of $\vec{q}$ and $\vec{d}$ or, equivalently, the cosine of the angle between $\vec{q}$ and $\vec{d}$.

Cosine for normalized vectors

- For normalized vectors, the cosine is equivalent to the dot product or scalar product.
- $\cos(\vec{q}, \vec{d}) = \vec{q} \cdot \vec{d} = \sum_i q_i \cdot d_i$
 - (if $\vec{q}$ and $\vec{d}$ are length-normalized).

Cosine: Example

How similar are these novels?

- SaS: Sense and Sensibility
- PaP: Pride and Prejudice
- WH: Wuthering Heights

term frequencies (counts)

term	SaS	PaP	WH
AFFECTION	115	58	20
JEALOUS	10	7	11
GOSSIP	2	0	6
WUTHERING	0	0	38

Cosine: Example

term frequencies (counts)

term	SaS	PaP	WH
AFFECTION	115	58	20
JEALOUS	10	7	11
GOSSIP	2	0	6
WUTHERING	0	0	38

log frequency weighting

term	SaS	PaP	WH
AFFECTION	3.06	2.76	2.30
JEALOUS	2.0	1.85	2.04
GOSSIP	1.30	0	1.78
WUTHERING	0	0	2.58

To simplify this example, we don't do idf weighting.

Cosine: Example

log frequency weighting

term	SaS	PaP	WH
AFFECTION	3.06	2.76	2.30
JEALOUS	2.0	1.85	2.04
GOSSIP	1.30	0	1.78
WUTHERING	0	0	2.58

log frequency weighting
& cosine normalization

term	SaS	PaP	WH
AFFECTION	0.789	0.832	0.524
JEALOUS	0.515	0.555	0.465
GOSSIP	0.335	0.0	0.405
WUTHERING	0.0	0.0	0.588

- $\cos(\text{SaS}, \text{PaP}) \approx 0.789 * 0.832 + 0.515 * 0.555 + 0.335 * 0.0 + 0.0 * 0.0 \approx 0.94$.
- $\cos(\text{SaS}, \text{WH}) \approx 0.79$
- $\cos(\text{PaP}, \text{WH}) \approx 0.69$
- Note that doc vectors are normalized.

Computing the cosine score

COSINESCORE(q)

```
1  float Scores[ $N$ ] = 0
2  float Length[ $N$ ]
3  for each query term  $t$ 
4  do calculate  $w_{t,q}$  and fetch postings list for  $t$ 
5      for each pair( $d, tf_{t,d}$ ) in postings list
6      do Scores[ $d$ ] +=  $w_{t,d} \times w_{t,q}$ 
7  Read the array Length
8  for each  $d$ 
9  do Scores[ $d$ ] = Scores[ $d$ ] / Length[ $d$ ]
10 return Top  $K$  components of Scores[]
```

Components of tf-idf weighting

Term frequency		Document frequency		Normalization	
n (natural)	$tf_{t,d}$	n (no)	1	n (none)	1
l (logarithm)	$1 + \log(tf_{t,d})$	t (idf)	$\log \frac{N}{df_t}$	c (cosine)	$\frac{1}{\sqrt{w_1^2 + w_2^2 + \dots + w_M^2}}$
a (augmented)	$0.5 + \frac{0.5 \times tf_{t,d}}{\max_t(tf_{t,d})}$	p (prob idf)	$\max\{0, \log \frac{N-df_t}{df_t}\}$	u (pivoted unique)	$1/u$
b (boolean)	$\begin{cases} 1 & \text{if } tf_{t,d} > 0 \\ 0 & \text{otherwise} \end{cases}$			b (byte size)	$1/CharLength^\alpha$, $\alpha < 1$
L (log ave)	$\frac{1 + \log(tf_{t,d})}{1 + \log(\text{ave}_{t \in d}(tf_{t,d}))}$				

Best known combination of weighting options Default: no weighting

variants of tf-idf

- We often use **different weightings** for queries and documents.
- Notation: ddd.qqq
- Example: Inc.ltn
- document: logarithmic tf, no df weighting, cosine normalization
- query: logarithmic tf, idf, no normalization
- Example query: “best car insurance”
- Example document: “car insurance auto insurance”

tf-idf example: Inc.ltn

Query: "best car insurance". Document: "car insurance auto insurance".

word	query					document				product
	tf-raw	tf-wght	df	idf	weight	tf-raw	tf-wght	weight	n'lized	
auto	0	0	5000	2.3	0	1	1	1	0.52	0
best	1	1	50000	1.3	1.3	0	0	0	0	0
car	1	1	10000	2.0	2.0	1	1	1	0.52	1.04
insurance	1	1	1000	3.0	3.0	2	1.3	1.3	0.68	2.04

- tf-raw: raw (unweighted) term frequency
- tf-wght: logarithmically weighted term frequency
- weight: the final weight of the term in the query or document
- n'lized: document weights after cosine normalization
- product: the product of final query weight and final document weight

$$\sqrt{1^2 + 0^2 + 1^2 + 1.3^2} \approx 1.92$$

$$1/1.92 \approx 0.52$$

$$1.3/1.92 \approx 0.68$$

Final similarity score between query and document:

$$\sum_i w_{qi} \cdot w_{di} = 0 + 0 + 1.04 + 2.04 = 3.08$$

Summary: Ranked retrieval in the vector space model

- Represent the query as a weighted tf-idf vector
- Represent each document as a weighted tf-idf vector
- Compute the cosine similarity between the query vector and each document vector
- Rank documents with respect to the query
- Return the top K (e.g., $K = 10$) to the user