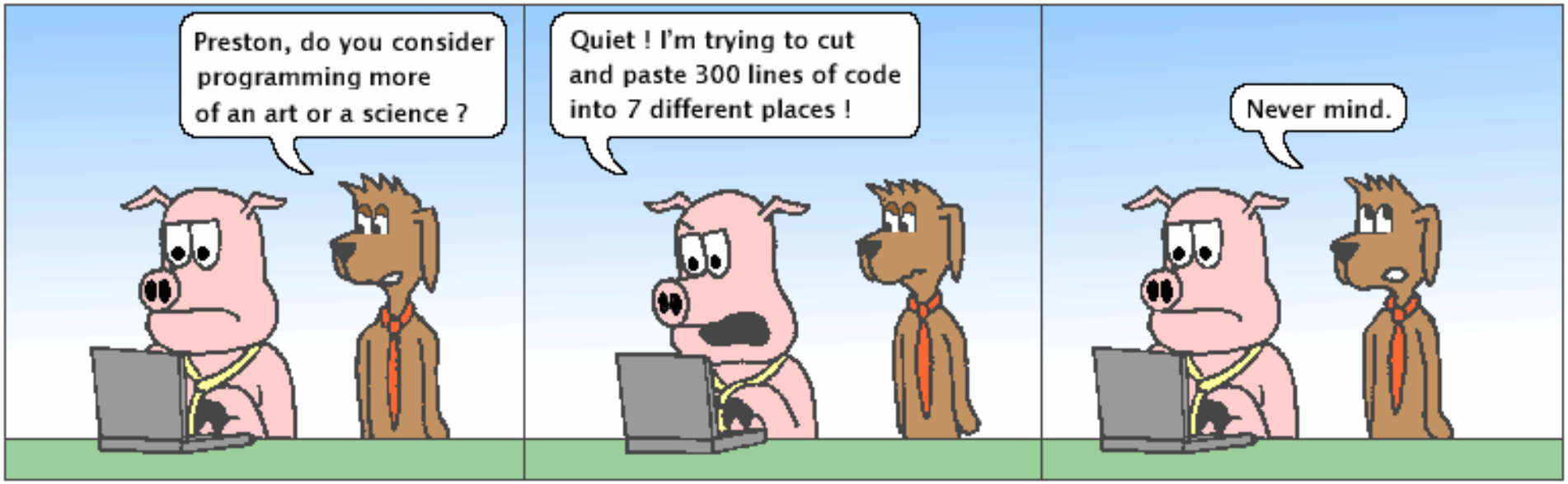
Formal Semantics of Programming Languages

"Program testing can be used to show the presence of bugs, but never to show their absence!" --Dijkstra

Hackles

By Drake Emko & Jen Brodzik



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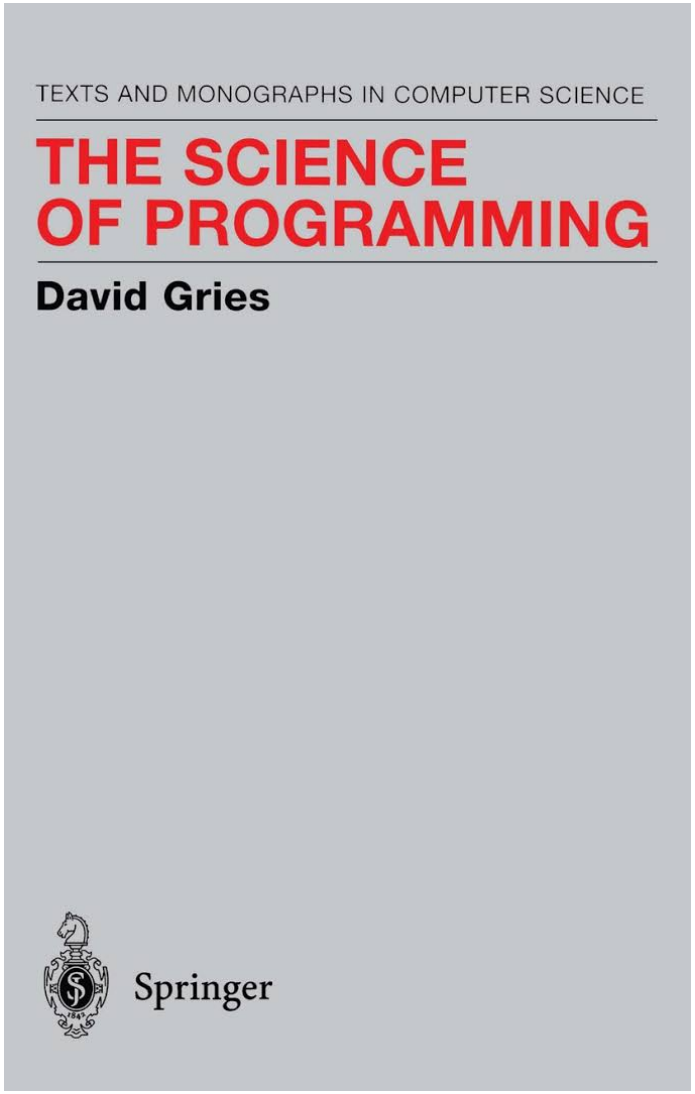

TEXTS AND MONOGRAPHS IN COMPUTER SCIENCE
THE SCIENCE OF PROGRAMMING
David Gries
Springer

Semantics of programming languages

- Basic components to describe programming languages
  - Syntax
  - Semantics
- Syntax is described by a grammar
  - a grammar is a 4-tuple (T,N,P,s)
    - T: a set of symbols (terminals)
    - N: a set of non-terminals
    - s ∈ N: starting non-terminal
    - P: a set of productions
      - a production has form: $\alpha \rightarrow \beta$ ($\alpha, \beta \in T \cup N$)
- There are many approaches to providing formal semantics to a programming language:
  - Operational
  - Denotational
  - Axiomatic
  - Algebraic

Algebraic specification of Stack and Queue

QUEUE

sorts: QUEUE, INT, BOOLEAN

operations:

new: --> QUEUE
add: QUEUE x INT --> QUEUE

empty: QUEUE --> BOOLEAN
del: QUEUE --> QUEUE
head: QUEUE --> INT U { error }

Semantics

empty(new()=true
emtpty(add(q,i))=false

del(New())=error
del(add(q,i))=if (empty(q)) then new() else
add(del(q),i)

head(new())=error
head(add(q,i))=if (empty(q)) then i
else head(q)

STACK

sorts: STACK, INT, BOOLEAN

operations:

new: --> STACK
push: STACK x INT --> STACK

empty: STACK --> BOOLEAN
pop: STACK --> STACK
top: STACK --> INT U { error }

Semantics

empty(new()) = true
empty(push(S, i)) = false

pop(new()) = error
pop(push(S, i)) = S

top(new()) = error
top(push(S,i)) = i

Axiomatic system

- An axiomatic system is any set of axioms from which some or all axioms can be used in conjunction to logically derive theorems.
  - E.g. Euclidean geometry
  - Axiom: accepted unproved statement
- It consists of
  - A grammar, i.e. a way of constructing well-formed formulae out of the symbols, such that it is possible to find a decision procedure for deciding whether a formula is a well-formed formula (wff) or not.
  - A set of axioms or axiom schemata: each axiom has to be a wff.
  - A set of inference rules.
  - A set of theorems. This set includes all the axioms, plus all wffs which can be derived from previously-derived theorems by means of rules of inference.
    - Unlike the grammar for wffs, there is no guarantee that there will be a decision procedure for deciding whether a given wff is a theorem or not.

The programming language

- A simple language:

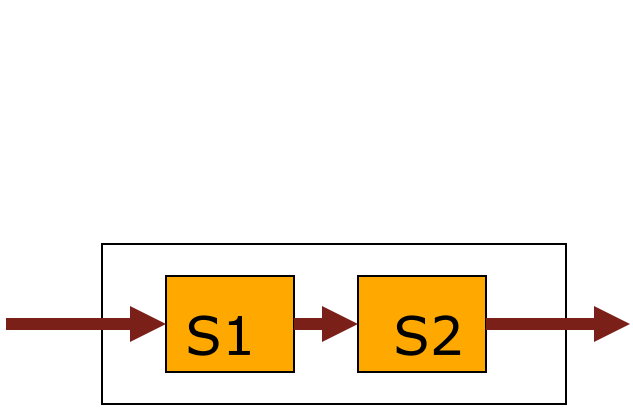
  W ::= V := T

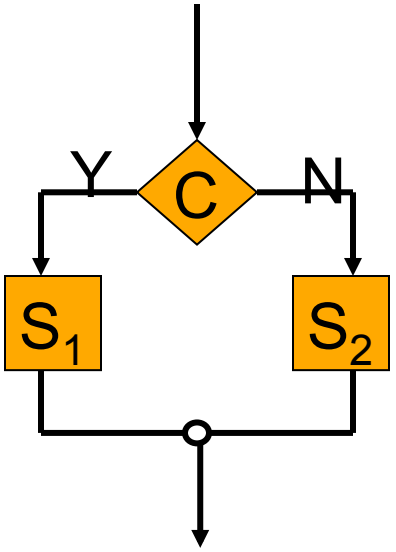
  W ::= if B then W else W

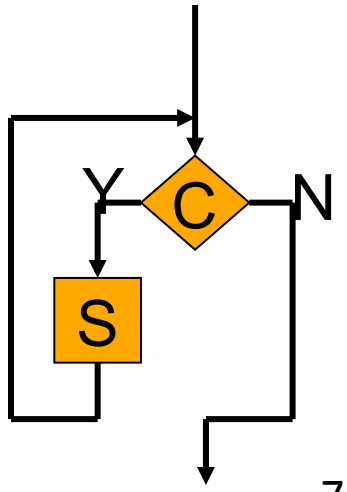
  W ::= while B do W

  W ::= W ; W

- An idealized, but nonetheless quite powerful, programming language.
- Remember that any program can be represented using these basic language constructs.







Hoare Logic

- We can use assertions to describe program semantics

  {x=0} x:=x+1 {x=1}

- Hoare Logic formalizes this idea

- An Hoare triple is in the following form:

  – {P} S {Q}

  where P and Q are assertions, and S is a program segment

- {P} S {Q} means "if we assume that P holds before S starts executing, then Q holds at the end of the execution of S"

  – I.e., if we *assume* P before execution of S, Q is *guaranteed* after execution of S

Example Hoare triples

- Whether the following triples are true? How can we prove?

  {x=0} x:=x+1 {x=1}

  {x+y=5} x:=x+5; y:=y-1 {x+y=9}

  {x+y=C} x:=x+5; y:=y-1 {x+y=C+4} where C is a place holder for any integer constant, i.e., it is equivalent to

  - ∀C, {x+y=C} x:=x+5; y:=y-1 {x+y=C+4}

  {x>C} x:=x+1 {x>C+1}

  {x>C} x:=x+1 {x>C}

  {x=1} x:=x+1 {x=1} incorrect

  {x+y=C} x:=x+1; y:=y-1 {x+y=C+1} incorrect

Proving properties of program segments

- How can we prove that:

  {x=0} x:=x+1 {x=1} is correct?

- We need an axiom which explains what assignment does
- First, we will need more notation
- We need to define the substitution operation
  - Let P[exp/x] denote the assertion obtained from P by replacing every appearance of x in P by the value of the expression exp
- Examples

  (x=0)[0/x]

  ≡ 0=0

  (x+y=z)[0/x]

  ≡ 0+y=z

  ≡ y=z

List of Axioms and rules

1) *Assignment axiom*

$$\{P[E/x]\}\ x := E\ \{P\}$$

2) *Consequence rule*

$$\frac{P \Rightarrow P',\ \{P'\}\ S\ \{Q\}}{\{P\}\ S\ \{Q\}} \qquad \frac{\{P\}\ S\ \{Q'\},\quad Q' \Rightarrow Q}{\{P\}\ S\ \{Q\}}$$

3) *Sequential composition rule*

$$\frac{\{P\}\ S1\ \{R\},\ \{R\}\ S2\ \{Q\}}{\{P\}\ S1;S2\ \{Q\}}$$

4) *If rule*

$$\frac{\{B \wedge P\}\ S1\ \{Q\}, \{\neg B \wedge P\}\ S2\ \{Q\}}{\{P\}\ \text{if B then S1 else S2}\ \{Q\}} \qquad \frac{\{B \wedge P\}\ S\ \{Q\},\ \neg B \wedge P \Rightarrow Q}{\{P\}\ \text{if } B \text{ then } S\ \{Q\}}$$

5) *While rule*

$$\frac{\{P \wedge B\}\ S\ \{P\}}{\{P\}\ \text{while B do S}\ \{\neg B \wedge P\}}$$

or combined with consequence rule we can have

$$\frac{\{P \wedge B\}\ S\ \{P\},\ \neg B \wedge P \Rightarrow Q}{\{P\}\ \text{while B do S}\ \{Q\}}$$

Axiom of Assignment

- Here is the **axiom of assignment**:

  {P[exp/x]} x:=exp {P}

  – where exp is a simple expression (no procedure calls in exp) that has no side effects (evaluating the expression does not change the state of the program)

- Now, let's try to prove

  {x=0} x:=x+1 {x=1}

  We have

  {x=1[x+1/x]} x:=x+1 {x=1} (by axiom of assignment)

  ≡ {x+1=1} x:=x+1 {x=1} (by definition of the substitution operation)

  ≡ {x=0} x:=x+1 {x=1} (by some axiom of arithmetic)

Axiom of Assignment

- Another example

  $\{x \geq 0\}\ x := x+1\ \{x \geq 1\}$

  We have

  $\{x \geq 1[x+1/x]\}\ x := x+1\ \{x \geq 1\}$ (by axiom of assignment)

  $\equiv \{x+1 \geq 1\}\ x := x+1\ \{x \geq 1\}$ (by definition of the substitution operation)

  $\equiv \{x \geq 0\}\ x := x+1\ \{x \geq 1\}$ (by some axiom of arithmetic)

Rules of Inference—rule of consequence

- Now we know

  {x=0} x:=x+1 {x=1}

- How can we prove

  {x=0} x:=x+1 {x>0}

- Once we prove a Hoare triple we may want to use it to prove other Hoare triples

- Here is the general rule (**rule of consequence 1**)

  – If {P}S{Q} and Q⇒Q' then we can conclude {P}S{Q'}

$$\frac{\{P\}\ S\ \{Q\},\ Q \Rightarrow Q'}{\{P\}\ S\ \{Q'\}}$$

- Example:

  {x=0} x:=x+1 {x=1} and x=1⇒x>0

  – hence, we conclude {x=0} x:=x+1 {x>0}

Rules of Inference—rule of consequence

- If we already proved {x≥0} x:=x+1 {x≥1}, then we should be able to conclude {x≥5} x:=x+1 {x≥1}
- Here is the general rule (**rule of consequence 2**)
  - If {P}S{Q} and P'⇒P then we can conclude {P'}S{Q}

$$\frac{\{P\}\ S\ \{Q\},\ P' \Rightarrow P}{\{P'\}\ S\ \{Q\}}$$

- Example
  - {x≥0} x:=x+1 {x≥1} and x≥5 ⇒ x≥0
  - hence, we conclude {x≥5} x:=x+1 {x≥1}

Rule of Sequential Composition

- Program segments can be formed by sequential composition
  - x:=x+5; y:=y-1 is sequential composition of two assignment statements x:=x+5 and y:=y-1
  - x:=x+5; y:=y-1; t:=0 is a sequential composition of the program segment x:=x +5; y:=y-1 and the assignment statement t:=0
- How do we reason about sequences of program statements?
- Here is the inference **rule of sequential composition**
  - If {P}S{Q} and {Q}T{R} then we can conclude that

    {P} S;T {R}

$$\frac{\{P\}\ S\ \{Q\}\ ,\ \{Q\}\ T\ \{R\}}{\{P\}\ S;T\ \{R\}}$$

Example: Swap

- prove a swap operation

  t:=x; x:=y; y:=t

- assume that x=A∧y=B holds before we start executing the swap segment.

- If swap is working correctly we would like x=B∧y=A to hold at the end of the swap (note that we did not restrict values A,B in any way)

  {x=A∧y=B} t:=x; x:=y; y:=t {x=B∧y=A}

- apply the axiom of assignment twice

  {x=B∧y=A [t/y]} y:=t {x=B∧y=A}

  ≡ {x=B∧t=A} y:=t {x=B∧y=A}

  {x=B∧t=A[y/x]} x:=y {x=B∧t=A}

  ≡ {y=B∧t=A} x:=y {x=B∧t=A}

Swap example

- Now since we have

  $\{y=B \wedge t=A\}\ x:=y\ \{x=B \wedge t=A\}$ and $\{x=B \wedge t=A\}\ y:=t\ \{x=B \wedge y=A\}$,

  – using the rule of sequential composition we get:

  $\{y=B \wedge t=A\}\ x:=y;\ y:=t\ \{x=B \wedge y=A\}$

- apply the axiom of assignment once more

  $\{y=B \wedge t=A[x/t]\}\ t:=x\ \{y=B \wedge t=A\}$

  $\equiv \{y=B \wedge x=A\}\quad t:=x\ \{y=B \wedge t=A\}$

- Using the rule of sequential composition once more

  $\{y=B \wedge x=A\}\ t:=x\ \{y=B \wedge t=A\}$ and $\{y=B \wedge t=A\}\ x:=y;\ y:=t\ \{x=B \wedge y=A\}$

  $\Rightarrow \{y=B \wedge x=A\}\ \ t:=x;\ x:=y;\ y:=t\ \ \{x=B \wedge y=A\}$

Inference rule for conditionals

- There are two inference rules for conditional statements, one for if-then and one for if-then-else statements
- For if-then-else statements the rule is (**rule of conditional 1)**

  If $\{P \wedge B\}\ S_1\ \{Q\}$ and $\{P \wedge \neg B\}\ S_2\ \{Q\}$ hold then we conclude that

  $\{P\}$ if B then S_1 else S_2 $\{Q\}$

$$\frac{\{P \wedge B\}\, S1\, \{Q\}, \{P \wedge \neg B\} S2 \{Q\}}{\{P\}\ \text{if B then S1 else S2}\ \{Q\}}$$

- For if-then statements the rule is (**rule of conditional 2)**

$$\frac{\{P \wedge B\}\, S\, \{Q\}, P \wedge \neg B \Rightarrow Q}{\{P\}\ \text{if B then S}\ \{Q\}}$$

Example for conditionals

- Here is an example

  if (x >y) then max:=x else max:=y

  – We want to prove

  {True} if (x>y) then max:=x else max:=y {max≥x∧max≥y}

{max≥x∧max≥y[x/max]} max:=x {max≥x∧max≥y} (Assignment axiom)

➔{x≥x∧x≥y} max:=x {max≥x∧max≥y} (definition of substitution)

➔ {True∧x≥y} max:=x {max≥x ∧ max≥y} (some axiom of arithmetics)

➔{x≥y} max:=x {max≥x ∧ max≥y} (some axiom of logic)

➔{x>y} max:=x {max≥x ∧ max≥y} (rule of consequence 2)

$$\frac{\{\mathrm{P} \wedge B\}\,\mathrm{S1}\,\{\mathrm{Q}\},\ \{\mathrm{P} \wedge \neg B\}S2\{Q\}}{\{\mathrm{P}\}\ \text{if B then S1 else S2}\ \{\mathrm{Q}\}}$$

Example for conditionals

{max≥x∧max≥y[y/max]} max:=y {max≥x∧max≥y} (r.assign.)

➔{y≥x∧y≥y} max:=y {max≥x∧max≥y} (definition of subs.)

➔{y≥x∧True} max:=y {max ≥x∧max≥y} (by arithmetics.)

➔{y≥x} max:=y {max≥x ∧ max≥y} (some axiom of logic)

➔{¬x>y} max:=y {max≥x∧max≥y} (some axiom of logic)

So we proved that

{ x>y} max:=x {max≥x∧max≥y}, and

{¬x>y} max:=y {max≥x∧max≥y}

$$\frac{\{P \wedge B\}\, S1\, \{Q\}, \{P \wedge \neg B\} S2 \{Q\}}{\{P\}\ \text{if B then S1 else S2}\ \{Q\}}$$

Then we can use the rule of conditional 1 and conclude that:

{True} if (x>y) max:=x else max:=y {max≥x∧max≥y}

Example for conditional rule 2

- Proof the following Hoare triple:

  {true} m:=y; if (x>y) m:=x; {m≥x ∧ m≥y}

  We need to prove {true} m:=y; {m=y}

  and {m=y} if (x>y) m:=x; {m≥x ∧ m≥y}

  To prove {m=y} if (x>y) m:=x; {m≥x ∧ m≥y},

  We need to show that

  1) {m=y ∧ x>y } m:=x; {m ≥ x ∧ m ≥ y}

  2) m=y ∧ NOT x>y ==> m ≥ x ∧ m ≥ y

  2) is true. (some properties of logic)

$$\frac{\{\mathrm{P} \wedge B\}\,\mathrm{S}\,\{\mathrm{Q}\},\ \mathrm{P} \wedge \neg B \Rightarrow Q}{\{\mathrm{P}\}\ \text{if B then S}\ \{\mathrm{Q}\}}$$

Example for conditional rule 2

To prove :

$\{m=y \wedge x>y\}\ m:=x;\ \{m \geq x \wedge m \geq y\}$

$\{m \geq x \wedge m \geq y\ [x/m]\}\ m:=x;\ \{m \geq x \wedge m \geq y\}$ by assignment axiom

➔

$\{x \geq x \wedge x \geq y\ [x/m]\}\ m:=x;\ \{m \geq x \wedge m \geq y\}$ by simplification

➔

$\{x \geq y\}\ m:=x;\ \{m \geq x \wedge m \geq y\}$ by simplification

Since $m = y \wedge x>y \Rightarrow x \geq y$; 3)

$\{m=y \wedge x>y\}\ m:=x;\ \{m \geq x \wedge m \geq y\}$ by consequence rule and 3)

What about the loops?

- Here is the inference rule (**rule of iteration**) for while loops

  If {P∧B} S {P} then we can conclude that {P} while B do S {¬B∧P }

$$\frac{\{\mathrm{P} \wedge B\}\ \mathrm{S}\ \{\mathrm{P}\}}{\{\mathrm{P}\}\ \text{while}\ \mathrm{B}\ \text{do}\ \mathrm{S}\ \{\neg B \wedge \mathrm{P}\}}$$

- This is what the inference rule for while loop is saying:
  - If you can show that every iteration of the loop preserves the property P,
  - and you know that the property holds before you start executing the loop,
  - then you can conclude that the property holds at the termination of the loop.
  - Also the loop condition will not hold at the termination of the loop (otherwise the loop would not terminate).

Loop invariants

- Given a loop
  - while B do S
  - Any assertion P which satisfies $\{P \wedge B\}\ S\ \{P\}$ is called a loop invariant
- A loop invariant is an assertion such that, every iteration of the loop body preserves it
  - In terms of Hoare triples this is equivalent to $\{P \wedge B\}\ S\ \{P\}$
- Note that rule of iteration given in the previous slide is for partial correctness
  - It does not guarantee that the loop will terminate

Using the rule of iteration

- To prove that a property Q holds after the loop while B do S terminates, we can use the following strategy
  - Find a strong enough loop invariant P such that:

    $(\neg B \wedge P) \Rightarrow Q$
  - Show that P is a loop invariant: $\{P \wedge B\}\ S\ \{P\}$
  - IF we can show that P is a loop invariant, we get

    $\{P\}$ while B do S $\{\neg B \wedge P\}$
  - Since we assumed that $(\neg B \wedge P) \Rightarrow Q$, using the rule of consequence 1, we get

    $\{P\}$ while B do S $\{Q\}$

The factorial example

```
{true}
  x := 0;  f := 1;
  while ( x != n ) do (x := x + 1; f := f * x;)
{f=n!}
```

Assume that $n \geq 0$. After computing

```
x := 0;  f := 1;
```

we have $f = x!$, i.e.,

```
{true} x := 0;  f := 1; {f=x!}
```

because it is true that 1 = 0!

We can show that:

$\{ f = x! \}$ `x := x + 1; f := f * x;` $\{ f = x! \}$

The factorial again... (2)

Now,

P is $f = x!$

B is $x \mathrel{!}= n$

¬ B is $x = n$

Using the inference rule for "while" loops:

{ $f = x!$ }
  while (x != n) do (x := x + 1; f := f * x;)
{ $f = x!$ & $x = n$}

The factorial again... (3)

Notice that

$f = x!$ & $x = n \Rightarrow f = n!$

This means two things:

- { true } x := 0; f := 1; { $f = x!$ }

AND

{ $f = x!$ } while (x != n) do (x := x + 1; f := f * x;)
{ f = n!}

Factorial (4)

In other words, the program establishes $f = n!$ without any preconditions on the initial values of f and n, assuming that we only deal with $n \geq 0$.

The rule for statement composition gives us:

```
{ true }    x := 0;  f := 1;
      while ( x != n )
          ( x := x + 1; f := f * x;)
{ f == n!}
```

So: this program does compute the factorial of n.

Factorial(5)

Our reasoning agrees with the intuition of loop invariants: we adjust some variables and make the invariant temporarily false, but we re-establish it by adjusting some other variables.

$\{ f = x! \}$ `x := x + 1;` $\{ f = (x - 1)! \}$

the invariant is "almost true"

$\{ f = (x - 1)! \}$ `f := f * x;` $\{ f = x! \}$

the invariant is back to normal

This reasoning is not valid for infinite loops:
the terminating condition P & ¬ B is never reached, and we know nothing of the situation following the loop.

Termination

- Proofs like these show only *partial correctness*.
  - Everything is fine if the loop stops.
  - Otherwise we don't know (but the program may be correct for most kinds of data).
- A reliable proof must show that all loops in the program are finite.
- We can prove termination by showing how each step brings us closer to the final condition.

The termination of factorial program for x=n!

- Informally
  - Initially, x = 0.
  - Every step increases x by 1, so we go through the numbers 0, 1, 2, ...
    - n >= 0 must be found among these numbers.
  - Notice that this reasoning will not work for n < 0
- The decreasing function
  - A loop terminates when the value of some function of program variables goes down to 0 during the execution of the loop.
  - For the factorial program, such a function could be n – x. Its value starts at n and decreases by 1 at every step.

Sum example (1)

- Consider the following program segment:

  sum:=0; i:=1; while (i <=10) do (sum:=sum+i; i:=i+1)

- We want to prove that $Q \equiv sum=\sum_{0\leq k \leq 10} k$

  holds at the loop termination, i.e., we want to prove the Hoare triple:

  {true} sum:=0; i:=1; while (i <=10) do (sum:=sum+i; i:=i+1) {Q}

  We need to find a strong enough loop invariant P

- Let's choose P as follows:

  $P \equiv i \leq 11 \wedge sum=\sum_{0\leq k<i} k$

Sum example (2)

- To use the rule of iteration we need to show $\{P \wedge B\}\ S\ \{P\}$ where

  $P \equiv i \leq 11 \wedge sum=\sum_{0 \leq k<i} k$

  S: sum:=sum+i; i:=i+1

  $B \equiv i \leq 10$

- Using the rule of assignment we get:

  $\{i \leq 11 \wedge sum=\sum_{0 \leq k<i} k\ [i+1/i]\}\ i:=i+1\ \{i \leq 11 \wedge sum=\sum_{0 \leq k<i} k\}$

  $\equiv\ \{i+1 \leq 11 \wedge sum=\sum_{0 \leq k<i+1} k\}\ i:=i+1\ \{i \leq 11 \wedge sum=\sum_{0 \leq k<i} k\}$

  $\equiv\ \{i \leq 10 \wedge sum=\sum_{0 \leq k<i+1} k\}\ i:=i+1\ \{i \leq 11 \wedge sum=\sum_{0 \leq k<i} k\}$

Sum example (3)

Using the rule of assignment one more time:

$\{i \le 10 \wedge sum = \sum_{0 \le k < i+1} k[sum+i/sum]\}$ sum:=sum+I $\{i \le 10 \wedge sum = \sum_{0 \le k < i+1} k\}$

$\equiv$ $\{i \le 10 \wedge sum+i = \sum_{0 \le k < i+1} k\}$ sum:=sum+i $\{i \le 10 \wedge sum = \sum_{0 \le k < i+1} k\}$

$\equiv$ $\{i \le 10 \wedge sum = \sum_{0 \le k < i} k\}$ sum:=sum+i $\{i \le 10 \wedge sum = \sum_{0 \le k < i+1} k\}$

Using the rule of sequential composition we get:

$\{i \le 10 \wedge sum = \sum_{0 \le k < i} k\}$ sum:=sum+i; i:=i+1 $\{i \le 11 \wedge sum = \sum_{0 \le k < i} k\}$

Sum example (4)

- Note that

$P \wedge B \equiv (i \leq 11 \wedge sum=\sum_{0\leq k<i}k) \wedge (i \leq 10)$

$\equiv i \leq 10 \wedge sum=\sum_{0\leq k<i}k$

$P \wedge \neg B \equiv (i \leq 11 \wedge sum=\sum_{0\leq k<i}k) \wedge \neg(i \leq 10)$

$\equiv i \leq 11 \wedge i > 10 \wedge sum=\sum_{0\leq k<i}k \equiv i = 11 \wedge sum=\sum_{0\leq k<i}k$

$\equiv sum=\sum_{0\leq k<11}k$

- Using the rule of iteration we get:

$\{i \leq 11 \wedge sum=\Sigma_{0\leq k<i}k\}$ while (i <=10) do (sum:=sum+i; i:=i+1)
$\{sum=\Sigma_{0\leq k<11}k\}$

Sum example (5)

- To finish the proof, apply assignment axiom

$\{i \le 11 \wedge sum = \sum_{0 \le k < i} k[1/i]\}\ i := 1\ \{i \le 11 \wedge sum = \sum_{0 \le k < i} k\}$

$\equiv\ \{1 \le 11 \wedge sum = \sum_{0 \le k < 1} k\}\ i := 1\ \{i \le 11 \wedge sum = \sum_{0 \le k < i} k\}$

$\equiv\ \{\quad sum = 0\}\quad i := 1\ \{i \le 11 \wedge sum = \sum_{0 \le k < i} k\}$

Another rule of assignment application

$\{sum = 0\ [0/sum]\}\ sum := 0\ \{sum = 0\}$

$\{0 = 0\}\ sum := 0\ \{sum = 0\}$

$\{true\}\ sum := 0\ \{sum = 0\}$

Sum example (6)

- Finally, combining the previous results with rule of sequential composition we get:

$\{true\}$

sum:=0; i:=1; while (i <=10) do (sum:=sum+i; i:=i+1)

$\{sum=\sum_{0\leq k \leq 10} k\}$

Difficulties in Proving Programs Correct

- Finding a loop invariant that is strong enough to prove the property we are interested in can be difficult
- Also, note that we did not prove that the loop will terminate
  - To prove total correctness we also have to prove that the loop terminates
- Things get more complicated when there are procedures and recursion

Difficulties in Proving Programs Correct

- Hoare Logic is a formalism for reasoning about correctness about programs
- Developing proof of correctness using this formalism is another issue
- In general proving correctness about programs is uncomputable
  - For example determining that a program terminates is uncomputable
- This means that there is no automatic way of generating these proofs
- Still Hoare's formalism is useful for reasoning about programs

- “I did not realize that the success of tests is that they test the programmer, not the program.”
  - C.A.R. Hoare, 2009, CACM